2.1a Taylor Expansion of $V(x)$ around $x=0$

$$V(x) \approx V(0) + \left.\frac{dV}{dx}\right|_{x=0}(x-0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x=0}(x-0)^2 + \frac{1}{3!}\left.\frac{d^3V}{dx^3}\right|_{x=0}(x-0)^3 + \frac{1}{4!}\left.\frac{d^4V}{dx^4}\right|_{x=0}(x-0)^4$$

$$= V(0) + \left.\frac{dV}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x=0} x^2 + \frac{1}{6}\left.\frac{d^3V}{dx^3}\right|_{x=0} x^3 + \frac{1}{24}\left.\frac{d^4V}{dx^4}\right|_{x=0} x^4$$

Given $V(0)=0$ and minimum of V is at $x=0$, $\left.\frac{dV}{dx}\right|_{x=0}=0$ as well.

$$= \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x=0} x^2 + \frac{1}{6}\left.\frac{d^3V}{dx^3}\right|_{x=0} x^3 + \frac{1}{24}\left.\frac{d^4V}{dx^4}\right|_{x=0} x^4.$$

$$= \frac{1}{2}kx^2 + \frac{1}{6}\gamma x^3 + \frac{1}{24}bx^4$$

$\therefore$ $k = \left.\frac{d^2V}{dx^2}\right|_{x=0}$, $\gamma = \left.\frac{d^3V}{dx^3}\right|_{x=0}$, $b = \left.\frac{d^4V}{dx^4}\right|_{x=0}$ are related to the values of derivative of V at $x=0$.

b $\hat{H} = \hat{H}_0 + \hat{H}'$

$$E_0^{(1)} = \langle 0|\hat{H}'|0\rangle$$

0 as the integral is odd

$$= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{\hbar}x^2}\left(\frac{1}{6}\gamma x^3 + \frac{1}{24}bx^4\right)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{b}{24}\int_{-\infty}^{\infty} x^4 e^{-\frac{m\omega}{\hbar}x^2}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \frac{b}{24} \cdot \frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-\frac{5}{2}}$$

$$= \frac{b}{32}\left(\frac{\hbar}{m\omega}\right)^2$$

Supplementary notes:

$$\int_{-\infty}^{\infty} x^4 e^{-a^2x^2}\,dx$$

$$= \frac{1}{a^5}\int_{-\infty}^{\infty} y^4 e^{-y^2}\,dy.$$

$$= \frac{1}{a^5}\left[y^3\left(-\frac{e^{-y^2}}{2}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(3y^2)\left(-\frac{e^{-y^2}}{2}\right)dy\right]$$

$$= \frac{1}{a^5}\left(\frac{3}{2}\int_{-\infty}^{\infty} y^2 e^{-y^2}\,dy\right)$$

$$= \frac{1}{a^5}\left(\frac{3}{2}\left[y\left(-\frac{e^{-y^2}}{2}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\frac{e^{-y^2}}{2}\,dy\right]\right)$$

$$= \frac{3}{4a^5}\sqrt{\pi}$$

$$= \frac{3\sqrt{\pi}}{4}a^{-5}$$

sub $a = \sqrt{\frac{m\omega}{\hbar}}$

$$\Rightarrow \frac{3\sqrt{\pi}}{4}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^{-5}$$

$$= \frac{3\sqrt{\pi}}{4}\left(\frac{m\omega}{\hbar}\right)^{-\frac{5}{2}}$$

c

From SQ 7

we know $\langle m|x|n\rangle = \left(\delta_{n-1,m}\sqrt{n}+\delta_{n+1,m}\sqrt{n+1}\right)(2\alpha)^{-\frac{1}{2}}$

$\langle m|x^2|n\rangle = \left(\delta_{n,m}(2n+1)+\delta_{n-2,m}\sqrt{n(n-1)}+\delta_{n+2,m}\sqrt{(n+1)(n+2)}\right)(2\alpha)^{-1}$

$$\langle m|x^3|n\rangle$$

$$=\sum_{j=0}^{\infty}\langle m|x^2|j\rangle\langle j|x|n\rangle$$

$$=\sum_{j=0}^{\infty}\frac{1}{2\alpha}\left((2j+1)\delta_{m,j}+\sqrt{j(j-1)}\,\delta_{m,j-2}+\sqrt{(j+1)(j+2)}\,\delta_{m,j+2}\right)\cdot\frac{1}{\sqrt{2\alpha}}\left(\delta_{n-1,j}\sqrt{n}+\delta_{n+1,j}\sqrt{n+1}\right)$$

$$=(2\alpha)^{-\frac{3}{2}}\Big[\left((2n-1)\delta_{m,n-1}+\sqrt{(n-1)(n-2)}\,\delta_{m,n-3}+\sqrt{n(n+1)}\,\delta_{m,n+1}\right)\sqrt{n}$$
$$+\left(2n+3)\delta_{m,n+1}+\sqrt{n(n+1)}\,\delta_{m,n-1}+\sqrt{(n+2)(n+3)}\,\delta_{m,n+3}\right)\sqrt{n+1}\Big]$$

$$=(2\alpha)^{-\frac{3}{2}}\left[\sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3}+3(n+1)^{\frac{3}{2}}\delta_{m,n+1}+3n^{\frac{3}{2}}\delta_{m,n-1}+\sqrt{n(n-1)(n-2)}\,\delta_{m,n-3}\right]$$

$$\langle m|x^4|n\rangle$$

$$=\sum^{\infty}\langle m|x^2|j\rangle\langle j|x^2|n\rangle$$

$$=\sum_j^{\infty}(2\alpha)^{-2}\left((2j+1)\delta_{m,j}+\sqrt{j(j-1)}\,\delta_{m,j-2}+\sqrt{(j+1)(j+2)}\,\delta_{m,j+2}\right)\left((2n+1)\delta_{j,n}+\sqrt{n(n-1)}\,\delta_{j,n-2}+\sqrt{(n+1)(n+2)}\,\delta_{j,n+2}\right)$$

$$=(2\alpha)^{-2}\Big[\left((2n+1)\delta_{m,n}+\sqrt{n(n-1)}\,\delta_{m,n-2}+\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right)(2n+1)$$
$$+\left((2n-3)\delta_{m,n-2}+\sqrt{(n-2)(n-3)}\,\delta_{m,n-4}+\sqrt{(n-1)n}\,\delta_{m,n}\right)\sqrt{n(n-1)}$$
$$+\left((2n+5)\delta_{m,n+2}+\sqrt{(n+2)(n+1)}\,\delta_{m,n}+\sqrt{(n+3)(n+4)}\,\delta_{m,n+4}\right)\sqrt{(n+1)(n+2)}\Big]$$

$$=(2\alpha)^{-2}\Big(3(n^2+(n+1)^2)\delta_{m,n}+2(2n-1)\sqrt{n(n-1)}\,\delta_{m,n-2}+2(2n+3)\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}$$
$$+\sqrt{n(n-1)(n-2)(n-3)}\,\delta_{m,n-4}+\sqrt{(n+1)(n+2)(n+3)(n+4)}\,\delta_{m,n+4}\Big)$$

Notice the δ functions, for $\langle x^3\rangle$, state n can only 'jump' to $n-3, n-1$, $n+1, n+3$ only, which is intuitive if we treat $\hat{x}$ as jumping up one state or down one state. Same for $\langle x^4\rangle$.

2.2a $E_n^{(1)} = \int \psi_n^* \hat{H}' \psi_n \, dx$

$= \langle n | \hat{H}' | n \rangle$

$= \frac{b}{24} \langle n | x^4 | n \rangle$

$= \frac{b}{24} \, 3\left(n^2 + (n+1)^2\right) \frac{1}{4\alpha^2}$

$= \frac{b}{32\alpha^2}\left(n^2 + (n+1)^2\right)$

From 2.1

$(2\alpha)^{-2}\Big(3\left(n^2+(n+1)^2\right)\delta_{m,n} + 2(2n-1)\sqrt{n(n-1)}\,\delta_{m,n-2} + 2(2n+3)\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}$

$+ \sqrt{n(n-1)(n-2)(n-3)}\,\delta_{m,n-4} + \sqrt{(n+1)(n+2)(n+3)(n+4)}\,\delta_{m,n+4} \Big)$

sub $m=n$, only non-zero term: $(2\alpha)^{-2}\left(3\left(n^2+(n+1)^2\right)\right)$

$\psi_0^{(1)} = \sum_{i=1}^{\infty} \frac{\langle i | \hat{H}' | 0 \rangle}{E_0^{(0)} - E_i^{(0)}} \psi_i^{(0)}$

$= \sum_{i=1}^{\infty} \frac{\langle i | \hat{H}' | 0 \rangle}{i\hbar\omega} \psi_i^{(0)}$

Notice that given $n=0$,

$\langle i | \hat{H}' | 0 \rangle \neq 0$ if $i = 2, 4$ only

$\because$ Each term is inversely proportional to i

$\therefore$ leading order term is $i=2$.

$\psi_0^{(1)} \approx \frac{\langle 2 | \hat{H}' | 0 \rangle}{2\hbar\omega} \psi_2^{(0)}$

$= \frac{6\sqrt{2}}{2\hbar\omega(4\alpha^2)} \frac{b}{24} \psi_2^{(0)}$

$= \frac{b\sqrt{2}}{32\hbar\omega\alpha^2} \psi_2^{(0)}$

2.3a $E_n^{(1)} = \langle n | H' | n \rangle$

$= e\mathcal{E} \langle n | x | n \rangle$

$= 0$

b $E_n^{(2)} = \sum_{i \neq n}^{\infty} \frac{|\langle i | H' | n \rangle|^2}{E_n - E_i}$

$= \sum_{i \neq n}^{\infty} e^2\mathcal{E}^2 \frac{|\langle i | x | n \rangle|^2}{(n-i)\hbar\omega}$

$= \sum_{i \neq n}^{\infty} \frac{e^2\mathcal{E}^2}{\hbar\omega} \frac{(\delta_{i,n-1}\sqrt{n} + \delta_{i,n+1}\sqrt{n+1})^2}{(n-i)\,2\alpha}$

$= \frac{e^2\mathcal{E}^2}{2m\omega^2} \left(\frac{n}{1} + \frac{n+1}{-1} \right)$

$= -\frac{e^2\mathcal{E}^2}{2m\omega^2}$ which is a constant for any n.

$E_n \approx E_n^{(0)} + E_n^{(2)}$

$= (n+\frac{1}{2})\hbar\omega - \frac{e\mathcal{E}}{2m\omega^2}$

c $\psi_n^{(1)} = \sum_{i \neq n}^{\infty} \frac{\langle i | H' | n \rangle}{E_n^{(0)} - E_i^{(0)}} \psi_i^{(0)}$

$= \sum_{i \neq n}^{\infty} \frac{e\mathcal{E}}{\hbar\omega} \frac{\delta_{i,n-1}\sqrt{n} + \delta_{i,n+1}\sqrt{n+1}}{\sqrt{2\alpha}\,(n-i)} \psi_i^{(0)}$

$= \frac{e\mathcal{E}}{\hbar\omega} \left(-\sqrt{\frac{n}{2\alpha}}\, \psi_{n-1}^{(0)} + \sqrt{\frac{n+1}{2\alpha}}\, \psi_{n+1}^{(0)} \right)$

First order perturbation to wave function 'mixes' the original $|n\rangle$ with $|n-1\rangle$ and $|n+1\rangle$

d $(x+b)^2 + c$

$= x^2 + 2bx + b^2 + c$

$= x^2 + ax$

$\therefore$ $b = \frac{a}{2}$, $c = -b^2$

$= -\frac{a^2}{4}$

e $\frac{1}{2} m\omega^2 \left(x^2 + \frac{2e\mathcal{E}}{m\omega^2} x \right)$

Take $x' = x + \frac{e\mathcal{E}}{m\omega^2}$ $\quad dx' = dx$.

$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 \left(x^2 + \frac{2e\mathcal{E}}{m\omega^2} x \right)$

$= -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 \left(\left(x + \frac{e\mathcal{E}}{m\omega^2}\right)^2 - \frac{e^2\mathcal{E}^2}{m^2\omega^4} \right)$

$= \underbrace{-\frac{\hbar}{2m} \frac{\partial^2}{\partial x'^2} + \frac{1}{2} m\omega^2 x'^2}_{H_0} - \underbrace{\frac{e^2\mathcal{E}^2}{2m\omega^2}}_{\text{constant.}}$

(since x' is just a variable)

$E_n = \langle n | \hat{H} | n \rangle$

$= \langle n | H_0 | n \rangle - \frac{e^2\mathcal{E}^2}{2m\omega^2} \langle n | n \rangle$

$= E_n - \frac{e^2\mathcal{E}^2}{2m\omega^2}$

f. We see that the perturbation and exact solutions have same energy. From the perturbation theory, we observe that $E_n^{(m)}$ contains product of m $\frac{\langle v \rangle}{\Delta E}$ with respect to different states. $\therefore E_n^{(m)} \propto \left(\frac{e\varepsilon}{m\omega^2}\right)^m$ However we saw that there are actually no contribution other than $m=2$ comparing to the exact solution.

If we compute perturbations up to infinite order, the solution would be exact but since only second order contribution is non-zero, 2^{nd} order alone will give exact result for this problem.

2.4a Recall $\psi_0(x)$ in 1D $= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$

$$\Psi = \psi_0(x)\psi_0(y)$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

b $E_x = (m+\frac{1}{2})\hbar\omega$

$E_y = (n+\frac{1}{2})\hbar\omega$ $\quad m, n = 0, 1, 2 \ldots$

$\therefore$ Second lowest energy $= (1+\frac{1}{2}+0+\frac{1}{2})\hbar\omega$
$= 2\hbar\omega$

Wave functions correspond to this energy: $\psi_1(x)\psi_0(y)$ or $\psi_0(x)\psi_1(y)$

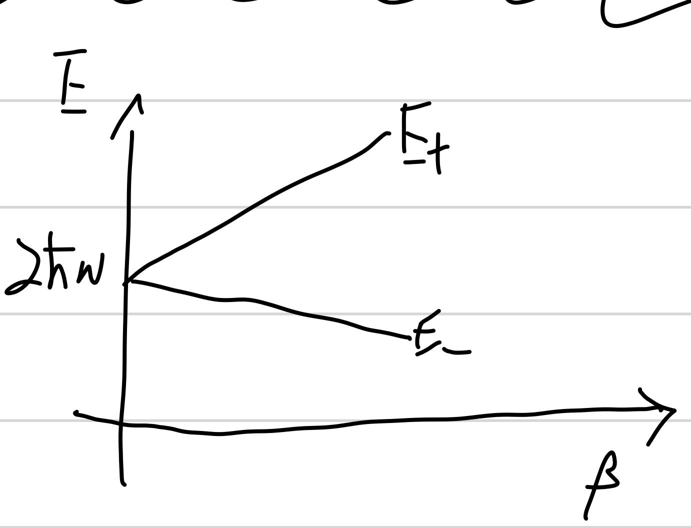
c
$$\begin{vmatrix} H_{nn}-E & H'_{ni} \\ H'_{in} & H_{ii}-E \end{vmatrix} = 0$$
Denote the wavefunctionas $|m,n\rangle$

$$H_{nn} = \langle 1,0|H|1,0\rangle$$
$$= E + \beta\langle 1|x|1\rangle\langle 0|y|0\rangle$$
$$= E_2 = 2\hbar\omega$$

$$H_{ii} = \langle 0,1|H|0,1\rangle$$
$$= E + \beta\langle 0|x|0\rangle\langle 1|y|1\rangle$$
$$= 2\hbar\omega$$

$$H'_{in} = \langle 0,1|H|1,0\rangle$$
$$= \beta\langle 0|x|1\rangle\langle 1|y|0\rangle$$
$$= \frac{\beta}{2\alpha} = \frac{\beta\hbar}{2m\omega}$$

$$H'_{ni} = H'^{*}_{in}$$
$$= \frac{\beta\hbar}{2m\omega}$$



The gap increases with β.

$$\begin{vmatrix} E_2-E & \frac{\beta\hbar}{2m\omega} \\ \frac{\beta\hbar}{2m\omega} & E_2-E \end{vmatrix} = 0$$

$$E_\pm = E_2 \pm \frac{\beta\hbar}{2m\omega}$$

For E_+, $\begin{pmatrix} -\frac{\beta\hbar}{2m\omega} & \frac{\beta\hbar}{2m\omega} \\ \frac{\beta\hbar}{2m\omega} & -\frac{\beta\hbar}{2m\omega} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ For E_-, $\begin{pmatrix} \frac{\beta\hbar}{2m\omega} & \frac{\beta\hbar}{2m\omega} \\ \frac{\beta\hbar}{2m\omega} & \frac{\beta\hbar}{2m\omega} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\therefore E_+ \Rightarrow \frac{1}{\sqrt{2}}\left(\psi_1(x)\psi_0(y) + \psi_0(x)\psi_1(y)\right)$

$E_- \Rightarrow \frac{1}{\sqrt{2}}\left(-\psi_0(x)\psi_1(y) + \psi_1(x)\psi_0(y)\right)$